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Weyl group invariants
– the case of projective unitary group $PU(p)$ –

芝浦工業大学・システム理工学部　亀子　正喜 (MASAKI KAMEKO)
COLLEGE OF SYSTEMS ENGINEERING AND SCIENCE,
SHIBAURA INSTITUTE OF TECHNOLOGY

岡山大学・名誉教授　三村　護 (MAMORU MIMURA)
OKAYAMA UNIVERSITY

1 Introduction

Let p be an odd prime. Let G be a compact connected Lie group. Let T be a maximal torus of G. We denote by W the Weyl group $N_G(T)/T$ of G. We write $H^*(X)$ for the mod p cohomology of a space X. Then, the Weyl group W acts on G, T, G/T, BG, BT and their cohomologies through the inner automorphism. The mod p cohomology of BT is a polynomial algebra $\mathbb{Z}/p[t_1, \ldots, t_n]$. We denote by $H^*(BT)^W$ the ring of invariants of the Weyl group W. Since G is path connected, the action of the Weyl group on BG is homotopically trivial and so the action of the Weyl group on the mod p cohomology $H^*(BG)$ is trivial. Therefore, we have the induced homomorphism

$$\eta^* : H^*(BG) \to H^*(BT)^W.$$

If $H_*(G; \mathbb{Z})$ has no p-torsion, the induced homomorphism η^* is an isomorphism. In [8], [9], Toda proved that even if $H_*(G; \mathbb{Z})$ has p-torsion, the induced homomorphism η^* is an epimorphism for $(G, p) = (F_4, 3)$, $(E_6, 3)$. However, Toda's results depend on the computation of the invariants. The purpose of this paper is not only to show the following Theorem 1.1 but also to give a proof without explicit computation of the Weyl group invariants.

We denote by y_2 a generator of $H^2(BG)$ for $(G,p) = (PU(p),p)$. Let Q_i be the Milnor operation defined by $Q_0 = \beta$, $Q_1 = \wp^1\beta - \beta\wp^1$, $Q_2 = \wp^p Q_1 - Q_1\wp^p$, ..., where $\wp^i$ is the i-th Steenrod reduced power operation. Let $y_{2p+2} = Q_0 Q_1 y_2$. For a graded vector space M, we denote by M^{even}, M^{odd} for graded subspaces of M spanned by even degree elements and odd degree elements, respectively. The following Theorems 1.1 and 1.2 are our results.

Theorem 1.1 *Let p be an odd prime. For $(G,p) = (PU(p),p)$, the induced homomorphism η^* above is an epimorphism. Moreover, we have*

$$H^*(BT)^W = H^{even}(BG)/(y_{2p+2}).$$

Theorem 1.2 *Let p be an odd prime. For $(G,p) = (F_4,3)$, $(E_6,3)$, $(E_7,3)$ and $(E_8,5)$, the induced homomorphism η^* above is an epimorphism.*

If G is a simply-connected, simple, compact connected Lie group, then G is one of classical groups $SU(n)$, $Sp(n)$ and Spin(n) or one of exceptional groups G_2, F_4, E_6, E_7, E_8. Since $H_*(G;\mathbb{Z})$ has no p-torsion except for the cases $(G,p) = (F_4,3)$, $(E_6,3)$, $(E_7,3)$, $(E_8,3)$ and $(E_8,5)$, the above theorem provides a supporting evidence for the following conjecture.

Conjecture 1.3 *Let p be an odd prime. Let G be a simply-connected, simple, compact connected Lie group. Then, the induced homomorphism η^* above is an epimorphism.*

To prove this conjecture, it remains to prove the case $(G,p) = (E_8,3)$. However, the mod 3 cohomology of BE_8 seems to be rather different from the other cases. For instance, the Rothenberg-Steenrod spectral sequence for the mod p cohomology for (G,p)'s in Theorems 1.1 and 1.2 collapses at the E_2-level but the one for the mod 3 cohomology of BE_8 is known not to collapse at the E_2-level and its computation is still an open problem. See [5].

In this paper, we prove Theorem 1.1. The proof in this paper is a restricted version of the proof in [3]. We will prove Theorems 1.1 and 1.2 both in [3] in the same manner.

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2 The Weyl group and the spectral sequence

As in § 1, let G be a compact connected Lie group. We consider the Leray-Serre spectral sequence associated with the fibre bundle

$$G/T \xrightarrow{\iota} BT \xrightarrow{\eta} BG.$$

Since BG is simply connected, the E_2-term is given by

$$H^*(BG) \otimes H^{*'}(G/T).$$

It converges to $gr\, H^*(BT)$. Moreover, the Weyl group acts on this spectral sequence and its action is given by

$$r^*(y \otimes x) = y \otimes r^* x,$$

where r is an element in W. Denote by σ the induced homomorphism $1 - r^*$. It is clear that

$$H^*(G/T)^W = \bigcap \operatorname{Ker} \sigma,$$

and $\sigma(x \otimes y) = x \otimes \sigma(y)$. Moreover, we have

$$(E_r^{*,*'})^W = \bigcap \operatorname{Ker} \sigma.$$

To relate the Weyl group invariants of $H^*(BT)$ and the one of E_∞-term, that is $gr\, H^*(BT)$, of the spectral sequence, we use the following lemma.

Lemma 2.1 *Suppose that $f : M \to N$ is a filtration preserving homomorphism of finite dimensional vector spaces with filtration. Denote by $grf : gr\, M \to gr\, N$ the induced homomorphism between associated graded vector spaces. Then, we have*

$$\dim \operatorname{Ker} grf \geq \dim \operatorname{Ker} f.$$

It is clear that

$$E_\infty^{*,0} = \operatorname{Im} \eta^* : H^*(BG) \to H^*(BT)^W,$$

so that $\dim E_\infty^{*,0} \leq \dim H^*(BT)^W$. By Lemma 2.1 above, we have

$$\sum_{*'} \dim (E_\infty^{*-*',*'})^W \geq \dim H^*(BT)^W.$$

Hence, if we have

$$(E_\infty^{*,*'})^W = E_\infty^{*,0},$$

we obtain

$$\dim H^*(BT)^W \leq \dim E_\infty^{*,0}$$

and the desired result $E_\infty^{*,0} = H^*(BT)^W$.

In [2], Kac mentioned the following theorem and Kitchloo gave the detail of Kac's result in §5 of [7].

Theorem 2.2 (Kac, Kitchloo) *Let p be an odd prime. Let G be a compact connected Lie group. Let T be a maximal torus of G and W the Weyl group of G. Then, we have $H^*(G/T)^W = H^0(G/T) = \mathbb{Z}/p$.*

Theorem 2.2 is the starting point of this paper. By Theorem 2.2, we have

$$(E_2^{*,*'})^W = (H^*(BG) \otimes H^{*'}(G/T))^W = (H^*(BG) \otimes \mathbb{Z}/p) = E_2^{*,0}.$$

Since the cohomology $H^*(G/T)$ has no odd degree generators, if $H_*(G;\mathbb{Z})$ has no p-torsion, then the E_2-term has no odd degree generators. Hence, it collapses at the E_2-level. Thus, we have that

$$(E_\infty^{*,*'})^W = E_\infty^{*,0} = H^*(BG).$$

Therefore, it is clear that the induced homomorphism $\eta^* : H^*(BG) \to H^*(BT)^W$ is an isomorphism if $H_*(G;\mathbb{Z})$ has no p-torsion.

However, for (G,p) in Theorems 1.1 and 1.2, $H_*(G;\mathbb{Z})$ has p-torsion and we have odd degree generators in the E_2-level. These odd degree generators do not survive to the E_∞-level. So, the spectral sequence does not collapse at the E_2-level. We deal with the spectral sequence for $(G,p) = (PU(p),p)$ in §4 and we will see that $(E_4^{*,*'})^W \neq E_4^{*,0}$. but still $(E_\infty^{*,*'})^W = E_\infty^{*,0}$ holds.

We end this section by recalling the mod p cohomology of G/T for $(G,p) = (PU(p),p)$.

Theorem 2.3 (Kac) *For $(G,p) = (PU(p),p)$, as an S-module, $H^*(G/T)$ is a free S-module generated by x_2^i $(0 \leq i \leq p-1)$, that is,*

$$H^*(G/T) = S\{x_2^i \mid 0 \leq i \leq p-1\},$$

where S is the image of the induced homomorphism $\iota^* : H^*(BT) \to H^*(G/T)$.

3 Cohomology of classifying spaces

In order to describe the odd degree generators of $H^*(BG)$, we consider non-toral elementary abelian p-subgroups of G. Non-toral elementary abelian p-subgroups of a compact connected Lie group G and their Weyl groups are described in [1] not only for (G,p) in Theorems 1.1 and 1.2 but also for $(G,p) = (E_8,3),(PU(p^n),p)$. For $(G,p) = (PU(p),p)$, there exists a unique maximal non-toral elementary abelian p-subgroup A of rank 2, up to conjugacy. Their Weyl groups $W(A) = N_G(A)/C_G(A)$ are also determined in [1]. We refer the reader to [1] for the detail.

From now on, we consider the case $(G,p) = (PU(p),p)$ only. We denote by $\xi : A \to G$ the inclusion of A into G and by abuse of notation, we denote the induced map $BA \to BG$ by the same symbol $\xi : BA \to BG$. It is easy to describe the ring of invariants $H^*(BA)^{W(A)}$ in terms of Dickson-Mui invariants because the Weyl groups $W(A)$ is $SL_2(\mathbb{Z}/p)$ and its action on $H^*(BA)$ is the obvious one.

We have

$$H^*(BA) = \mathbb{Z}/p[t_1,t_2] \otimes \bigwedge(dt_1,dt_2) = \mathbb{Z}/p[t_1,t_2]\{1,dt_1,dt_2,dt_1dt_2\},$$

where dt_i's are generators of $H^1(BA_2)$, $t_i = \beta dt_i$, and β is the Bockstein homomorphism. We denote the element dt_1dt_2 by u_2. We denote by e_2 the element $Q_0Q_1u_2$. Dickson invariants $c_{2,0}$, $c_{2,1}$ are defined by

$$\prod_{x\in\mathbb{Z}/p\{t_1,t_2\}} (X-x) = X^{p^2} - c_{2,1}X^p + c_{2,0}X.$$

Moreover, we have $c_{2,0} = e_2^{p-1}$. Then, the ring of invariants is given as follows:

$$H^*(BA)^{W(A)} = \mathbb{Z}/p[c_{2,1},e_2]\{1,Q_0u_2,Q_1u_2,u_2\}.$$

See [6] for the detail.

Let

$$N_0 = \mathbb{Z}/p[c_{2,1},e_2]\{1,Q_1u_2\},$$
$$N_1 = \mathbb{Z}/p[c_{2,1},e_2]\{Q_0u_2,u_2\}.$$

Since

$$Q_0u_2 \cdot Q_1u_2 = -e_2u_2,$$

it is easy to see the following proposition.

Proposition 3.1 *There exist short exact sequences*

(1) $\quad 0 \to N_0 \xrightarrow{Q_0u_2} N_1 \to N_1^{even}/(e_2) \to 0,$

(2) $\quad 0 \to N_1 \xrightarrow{Q_1} N_0 \to N_0^{even}/(e_2) \to 0.$

By comparing odd degree generators of $H^*(BG)$ and the image of the induced homomorphism $\xi^* : H^*(BG) \to H^*(BA)$, it is easy to see that

$$\xi^* : H^{odd}(BG) \to H^{odd}(BA)$$

is a monomomorphism and

$$\xi^* : H^{odd}(BG) \to H^{odd}(BA)^{W(A)}$$

is an isomorphism. For $H^*(BG)$, we refer the reader to [4].

Let y_2 be the generator of $H^2(BG)$ such that $\xi^*(y_2) = u_2$. Let $y_3 = Q_0 y_2$, $y_{2p+1} = Q_1 y_2$, $y_{2p+2} = Q_0 Q_1 y_2$ and choose y_{2p^2-2p} such that $\xi^*(y_{2p^2-2p}) = c_{2,1}$. We put

$$M_0 = \mathbb{Z}/p[y_{2p^2-2p}, y_{2p+2}]\{1, y_{2p+1}\},$$
$$M_1 = \mathbb{Z}/p[y_{2p^2-2p}, y_{2p+2}]\{y_3, y_2\}.$$

It is clear that $H^*(BG)$ is a $\mathbb{Z}/p[y_{2p^2-2p}, y_{2p+2}]$-module. For dimensional reasons, we have $Q_1 y_{2p^2-2p} = 0$. Thus, we have the following proposition.

Proposition 3.2 *There holds*

(1) $\quad \xi^* M_0 \oplus \xi^* M_1 = \operatorname{Im} \xi^*.$

Moreover, there exist the following short exact sequences:

(2) $\quad 0 \to M_0 \xrightarrow{y_3} M_1 \to M_1^{even}/(y_{2p+2}) \to 0,$

(3) $\quad 0 \to M_1 \xrightarrow{Q_1} M_0 \to M_0^{even}/(y_{2p+2}) \to 0.$

4 The spectral sequence

In this section, we prove Theorem 1.1 by computing the Leray-Serre spectral sequence for

$$G/T \xrightarrow{\iota} BT \xrightarrow{\eta} BG,$$

where $G = PU(p)$. The E_2-term of the spectral sequence is given by

$$E_2 = H^*(BG) \otimes H^{*'}(G/T)$$

as an $H^*(BG) \otimes S$-algebra. The algebra generator is $1 \otimes x_2$. So, the first non-trivial differential is determined by $d_r(1 \otimes x_2)$ for some $r \geq 2$.

Proposition 4.1 *For $r < 3$, $d_r = 0$. The first nontrivial differential is d_3 and there holds*

$$d_3(1 \otimes x_2) = \alpha(y_3 \otimes 1)$$

for some $\alpha \neq 0 \in \mathbb{Z}/p$.

Proof Suppose that $d_{r_0}(1 \otimes x_2) \neq 0$ for some $r_0 < 3$. Then, up to degree ≤ 2, E_{r_0+1}-term is generated by $1 \otimes 1$ as an $H^*(BG) \otimes S$-module. So, for $r_1 \geq r_0$, $\operatorname{Im} d_{r_1}$ does not contain any element of degree less than or equal to 3. Hence, $y_3 \otimes 1$ survive to the E_∞-term. Then, $\eta^*(y_3) \neq 0$. This contradicts the fact $E_\infty^{odd} = \{0\}$ since $\deg y_3 = 3$ is odd. Therefore, we have $d_r(1 \otimes x_2) = 0$ for $r < 3$.

Next, we verify that $d_3(1 \otimes x_2) = \alpha(y_3 \otimes 1)$ for some $\alpha \neq 0$ in $\mathbb{Z}/p$. If $\operatorname{Im} d_3$ does not contain $y_3 \otimes 1$, then up to degree ≤ 3, the spectral sequence collapses at the E_4-level and $y_3 \otimes 1$ survives to the E_∞-term. As in the above, it is a contradiction. Hence, the proposition holds. □

To consider the next nontrivial differential, first, we show the following lemmas.

Lemma 4.2 *Both*

(1) *the multiplication by y_3 and*

(2) *the multiplication by y_{2p+2}*

are zero on $\operatorname{Ker} \xi^*$.

Proof Suppose that $z \in \operatorname{Ker} \xi^*$.

Then, $\xi^*(z \cdot y_3) = 0$ and $\deg(z \cdot y_3)$ is odd. Hence, we have $z \cdot y_3 = 0$ in $H^*(BG)$.

We also get $Q_1(z \cdot y_3) = 0$. On the other hand, , since $\xi^*(Q_1 z) = 0$ and $\deg(Q_1 z)$ is odd, we have $Q_1 z = 0$ in $H^*(BG)$. Hence, we get

$$Q_1(z \cdot y_3) = Q_1 z \cdot y_3 - z \cdot y_{2p+2} = -z \cdot y_{2p+2} = 0.$$

So, we obtain $z \cdot y_{2p+2} = 0$. Thus, we have the desired result. □

Then, we may consider

$$E_3 = E_2 = (M_0 \oplus M_1 \oplus \operatorname{Ker} \xi^*) \otimes H^*(G/T),$$

as a $\mathbb{Z}/p[y_{2p^2-2p}, y_{2p+2}] \otimes S$-module. By Propositions 4.1 and 3.2 (2) and Lemma 4.2 (1), we have the E_4-term:

$$E_4 = (M_1 \otimes N_{p-1}) \oplus (M_1^{even}/(y_{2p+2}) \otimes N_{\leq p-2}) \oplus (M_0 \otimes N_0) \oplus (\operatorname{Ker} \xi^* \otimes H^*(G/T)),$$

where $N_{\le i}$ is the S-submodule of $H^*(G/T)$ generated by x_2^k $(k \le i)$ and N_i is the S-submodule generated by a single element x_2^i in $H^*(G/T)$. The above direct sum decomposition is in the category of $\mathbb{Z}/p[y_{2p^2-2p}, y_{2p+2}] \otimes S$-modules.

Now, we investigate the action of the Weyl group on the spectral sequence in terms of σ. Recall that $\sigma = 1 - r^*$, where $r \in W$. Then, σ acts on the spectral sequence by $\sigma(y \otimes x) = y \otimes \sigma(x)$ and it commutes with the differential d_r for $r \ge 2$.

Lemma 4.3 *There holds $\sigma(x_2^i) \in N_{\le i-1}$ for all σ.*

Proof Since d_3 commutes with σ, and since $\sigma(y_3 \otimes 1) = 0$, we have

$$d_3(\sigma(1 \otimes x_2)) = 0.$$

Suppose that $\sigma(x_2) = \beta x_2 + s$ for some $\beta \in \mathbb{Z}/p$ and s in S. Then, we have

$$d_3(\beta(1 \otimes x_2) + 1 \otimes s) = \alpha\beta(y_3 \otimes 1) = 0.$$

Therefore, we have $\beta = 0$ and $\sigma(x_2) \in N_0 = S$. In general, we have

$$\sigma(xy) = \sigma(x)y + x\sigma(y) - \sigma(x)\sigma(y).$$

Hence, we have

$$\sigma(x_2^i) = \sigma(x_2)x_2^{i-1} + x_2\sigma(x_2^{i-1}) - \sigma(x_2)\sigma(x_2^{i-1}) \in N_{\le i-1},$$

as desired. □

Remark 4.4 By Lemma 4.3, σ acts trivially on $N_i = N_{\le i}/N_{\le i-1}$. Hence, it is easy to see that

$$(E_4^{*,*'})^W = (M_1^{odd} \oplus y_{2p+2}M_1^{even}) \otimes N_{p-1} \oplus (M_1^{even}/(y_{2p+2}) \oplus M_0 \oplus \operatorname{Ker}\xi^*) \otimes \mathbb{Z}/p \ne E_4^{*,0}.$$

Now, we begin to compute the next nontrivial differential.

Proposition 4.5 *For $r \ge 4$ such that $E_r = E_4$, we have*

$$d_r(M_0 \otimes N_0) = d_r(\operatorname{Ker}\xi^* \otimes H^*(G/T)) = d_r(M_1^{even}/(y_{2p+2}) \otimes N_{\le p-2}) = \{0\}.$$

Proof Since $M_0 \otimes N_0$ is generated by $M_0 \otimes \mathbb{Z}/p$ as an $\mathbb{Z}/p[y_{2p^2-2p}, y_{2p+2}] \otimes S$-module, $d_r(M_0 \otimes N_0) = \{0\}$ holds for $r \ge 4$. For $M_1^{even}/(y_{2p+2}) \otimes N_{\le p-2}$, there exists no odd degree generators. Hence, we have

$$d_r(M_1^{even}/(y_{2p+2}) \otimes N_{\le p-2}) \subset E_4^{odd} = M_1^{odd} \otimes N_{p-1} \oplus M_0^{odd} \otimes N_0.$$

On the one hand, the multiplication by $y_{2p+2} \otimes 1$ is zero on $M_1^{even}/(y_{2p+2}) \otimes N_{\leq p-2}$. On the other hand, the multiplication by $y_{2p+2} \otimes 1$ is a monomorphism on $M_1^{odd} \otimes N_{p-1} \oplus M_0^{odd} \otimes N_0$. Hence, we have

$$d_r(M_1^{even}/(y_{2p+2}) \otimes N_{\leq p-2}) = \{0\}.$$

Finally, by Lemma 4.2, the same holds for $\operatorname{Ker} \xi^* \otimes H^*(G/T)$ and so we obtain

$$d_r(\operatorname{Ker} \xi^* \otimes H^*(G/T)) = \{0\}. \qquad \square$$

Next, we show the following proposition.

Proposition 4.6 *If $r \geq 4$ and if d_r is nontrivial, then $r \geq 2p-1$.*

Proof Suppose that we have a nontrivial differential d_r for some $r < 2p-1$, say,

$$d_r(z \otimes x_2^{p-1}) = z_{i_1} \otimes x_1' + \cdots + z_{i_\ell} \otimes x_\ell',$$

where $z \in M_1$, $1 \leq i_1 < \cdots < i_\ell \leq L$, $\{z_1, \ldots, z_L\}$ is a basis for

$$(M_1^{even}/(y_{2p+2}) \oplus M_0 \oplus \operatorname{Ker} \xi^*)^{\deg z + r},$$

and $x_1', \ldots, x_\ell' \in H^{2p-1+r}(G/T)$, $x_1', \ldots, x_\ell' \neq 0$. Since $H^*(G/T)^W = \mathbb{Z}/p$, for $x_1' \neq 0$ in $H^{2p-1+r}(G/T)$, there exists σ such that $\sigma(x_1') \neq 0$. Therefore, we have

$$\sigma d_r(z \otimes x_2^{p-1}) \neq 0.$$

On the other hand, by Lemma 4.3, we have $\sigma(x_2^{p-1}) \in N_{\leq p-2}$. Hence, by Proposition 4.5 above, we have

$$\sigma d_r(z \otimes x_2^{p-1}) \in d_r(M_1^{even}/(y_{2p+2}) \otimes N_{\leq p-2}) = \{0\}.$$

This is a contradiction. Hence, we have $r \geq 2p-1$. $\square$

Finally, we complete the computation of the spectral sequence.

Proposition 4.7 *There holds $d_{2p-1}(M_1 \otimes N_{p-1}) = (M_0^{odd} \oplus y_{2p+2} M_0^{even}) \otimes N_0$.*

Proof The E_{2p-1}-term is equal to

$$M_1 \otimes N_{p-1} \oplus M_1^{even}/(y_{2p+2}) \otimes N_{\leq p-2} \oplus M_0 \otimes N_0 \oplus (\operatorname{Ker} \xi^*) \otimes H^*(G/T)$$

and

$$d_{2p-1}(M_1^{even}/(y_{2p+2}) \otimes N_{\leq p-2} \oplus M_0 \otimes N_0 \oplus (\operatorname{Ker} \xi^*) \otimes H^*(G/T)) = \{0\}.$$

Since $M_1^{even}/(y_{2p+2}) \otimes N_{\leq p-2} \oplus M_0 \otimes N_0 \oplus (\mathrm{Ker}\,\xi^*) \otimes H^*(G/T)$ is generated by elements of the second degree less than $2p-2$, that is, the elements in $E_r^{*,*'}$ $(*' < 2p-2)$, it is clear that

$$d_r(M_1^{even}/(y_{2p+2}) \otimes N_{\leq p-2} \oplus M_0 \otimes N_0 \oplus (\mathrm{Ker}\,\xi^*) \otimes H^*(G/T)) = \{0\}$$

for all $r \geq 2p-1$.

On the other hand, since all elements in $(M_0^{odd} \oplus y_{2p+2}M_0^{even}) \otimes \mathbb{Z}/p$ do not survive to the E_∞-term and since $d_r(M_0 \otimes N_0) = \{0\}$ for all $r \geq 2$, all elements in $(M_0^{odd} \oplus y_{2p+2}M_0^{even}) \otimes \mathbb{Z}/p$ must be hit by nontrivial differentials.

Suppose that there exists an element in $(M_0^{odd} \oplus y_{2p+2}M_0^{even}) \otimes \mathbb{Z}/p$ that is not hit by d_{2p-1}. Let $z \otimes 1$ be a such element with the lowest degree s. Up to degree $< s$, by Proposition 3.2,

$$d_{2p-1} : M_1^i \otimes N_{p-1} \longrightarrow (M_0^{odd} \oplus y_{2p+2}M_0^{even})^{i+2p-1} \otimes N_0$$

is an isomorphism for $i < s$.

Then, $\mathrm{Ker}\,d_{2p-1}$ is equal to $M_1^{even}/(y_{2p+2}) \otimes N_{\leq p-2} \oplus M_0 \otimes N_0 \oplus (\mathrm{Ker}\,\xi^*) \otimes H^*(G/T)$ up to degree s. Therefore, for $r \geq 2p$, $\mathrm{Im}\,d_r = \{0\}$ up to degree $\leq s$. Hence the element $z \otimes 1$ survives to the E_∞-term. This is a contradiction. So, the proposition holds. $\square$

So, by Propositions 4.5 and 4.7, we have

$$E_{2p} = (M_1^{even}/(y_{2p+2}) \otimes N_{\leq p-2}) \oplus (M_0^{even}/(y_{2p+2}) \otimes N_0) \oplus (\mathrm{Ker}\,\xi^* \otimes H^*(G/T)).$$

Since there are no odd degree elements in the E_{2p}-term, the spectral sequence collapses at the E_{2p}-level and we obtain $E_\infty = E_{2p}$ and

$$(E_\infty^{*,*'})^W = E_\infty^{*,0} = (M_1^{even}/(y_{2p+2}) \oplus M_0^{even}/(y_{2p+2}) \oplus \mathrm{Ker}\,\xi^*) \otimes \mathbb{Z}/p.$$

This completes the proof of Theorem 1.1.

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